Networks

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Tom Johnson
75 rue de la Roquette
75011 Paris
TEL 33 1 43 48 90 57
email: tom@johnson.org
www.tom.johnson.org

I was pleased to learn that it would be possible to make a little exhibition of drawings here in Berlin, as well as presenting my lecture, because the *Networks* I'm working on now are as much visual as aural, and I think these structures need to be seen as well as heard. Of course, those who are simply reading this text will only see the drawings that are included here, but hopefully this will be enough to convey the general idea.

Many composers make graphs, tables, or charts of some sort to calculate the details of their music, and I have been doing this for a long time, but finding the system visually has become a particularly important part of the *Networks* I have been working on since 2005, which have mostly to do with harmony and with defining groups of chords. My earlier music also sometimes concerned chord groups and combination theory, the most obvious example being *The Chord Catalogue*, but this interest took a new turn when a young Dutch Composer, Samuel Vriezen, showed me how he had composed a group of 11 five-note chords such that each chord had two notes in common with each other chord, and a mathematician friend, Jean-Paul Allouche, suggested that I investigate "block designs." This is a relatively new kind of combination theory that I don't think Vriezen had ever studied either, and which, in fact, is not widely know even to mathematicians. The principles are rather simple, however, and after studying the subject a bit, I realized that these networks of subgroups or chords could lead me to rich and yet unknown musical materials.

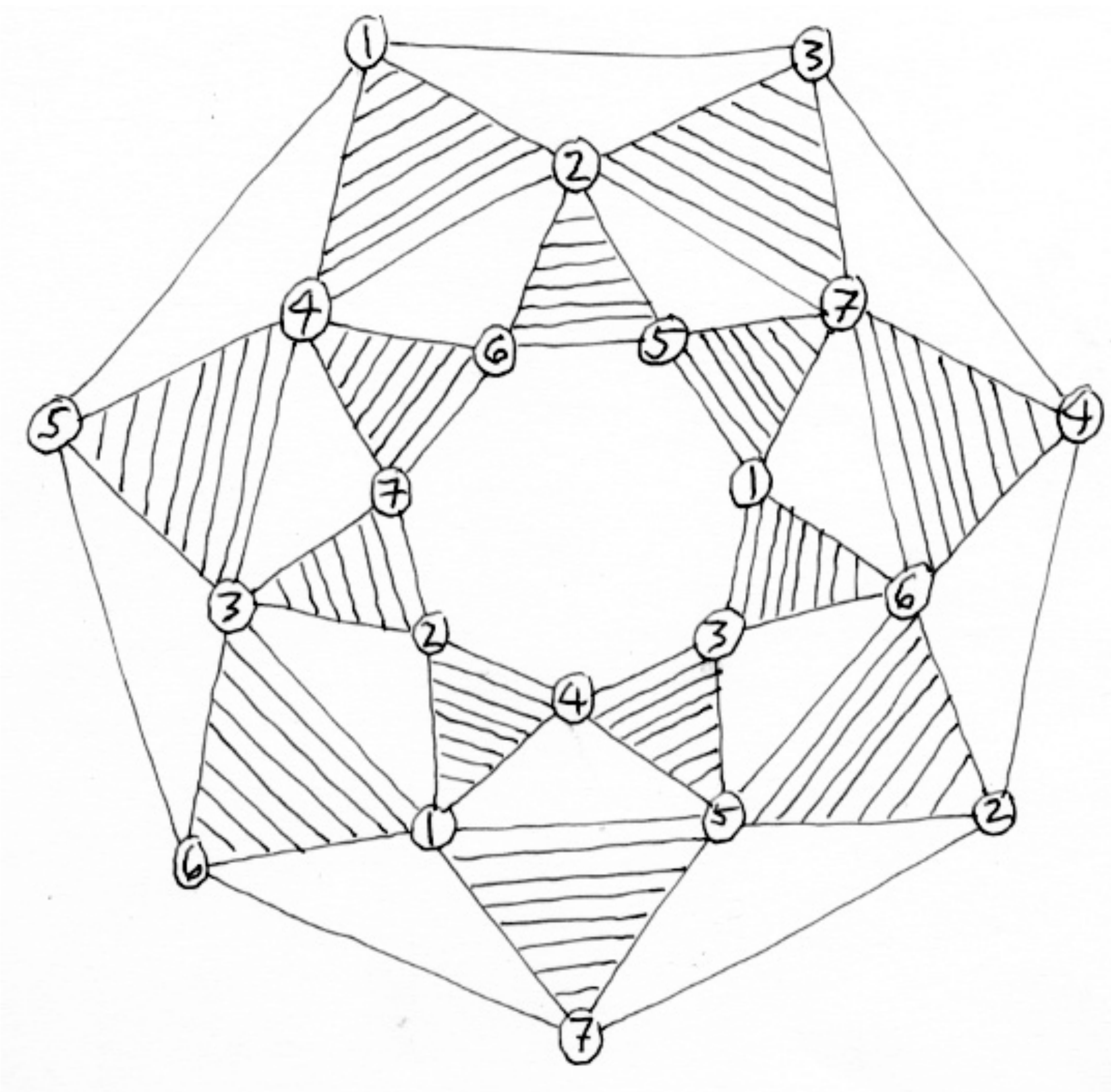
Let me begin with the example of a block design known as (7, 3, 1). That means that 7 notes (elements) are divided into chords (sub-groups) of 3 notes, in such a way that each pair of notes comes together in one chord. One can do it in this way:

(1,2,3), (3,4,7), (2,4,6), (2,5,7), (1,6,7), (3,5,6) (1,4,5),

but one can also solve the problem in this way:

(1,2,4), (2,3,7), (4,6,7), (2,5,6), (1,5,7), (1,3,6), (3,4,5).

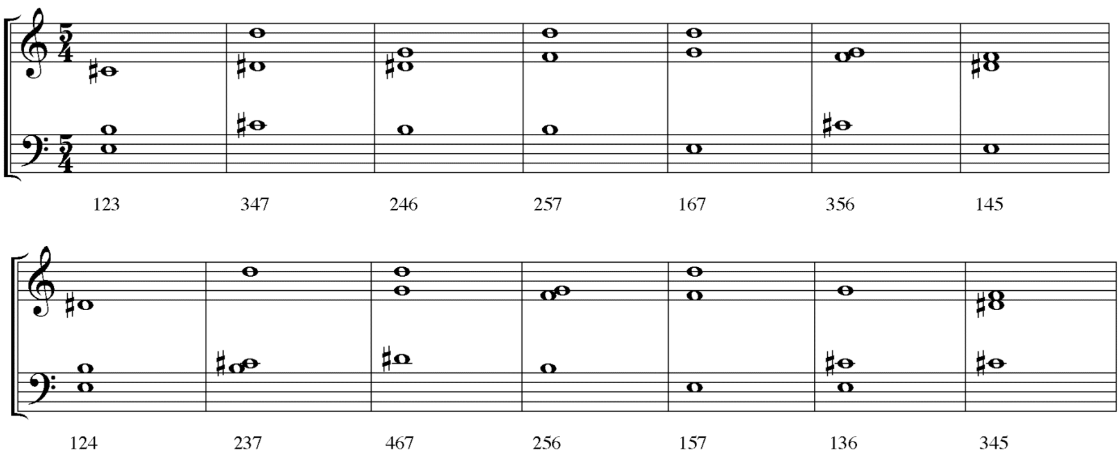
How can we see the relationships between these three-note chords? Is there a way to combine the two solutions? How do we find the beginning, the end, the continuity of the logic? How can we find the nerve of the system? Well, the answer to all these questions is to begin drawing pictures. In this way I found four different representations, all of which are in the exhibition, but I particularly want you to see the network in this way, where the white triangles represent the first solution, the gray triangles represent the second, and the 14 chords are all shown twice:



Of course, mathematicians have been studying such structures for a long time, and I was fortunate to make contact with two mathematicians from the University of Vermont who took an interest in my drawings and wrote some comments, which are posted along with the drawings in the exhibition space. About this particular drawing, Jeffrey Dinitz, who specializes in combinatorial designs, and Dan Archdeacon, who studies topological graph theory, sent me the following text:

This shows four representations of the universal coverings of K_7 (the complete graph on 7 vertices) on the torus. In each case the seven shaded triangles form one Fano plane (the projective plane of order 2) and the seven white triangles form another. One can see here how (7,3,2) combines two (7,3,1) systems, one of white triangles and one of shaded triangles.

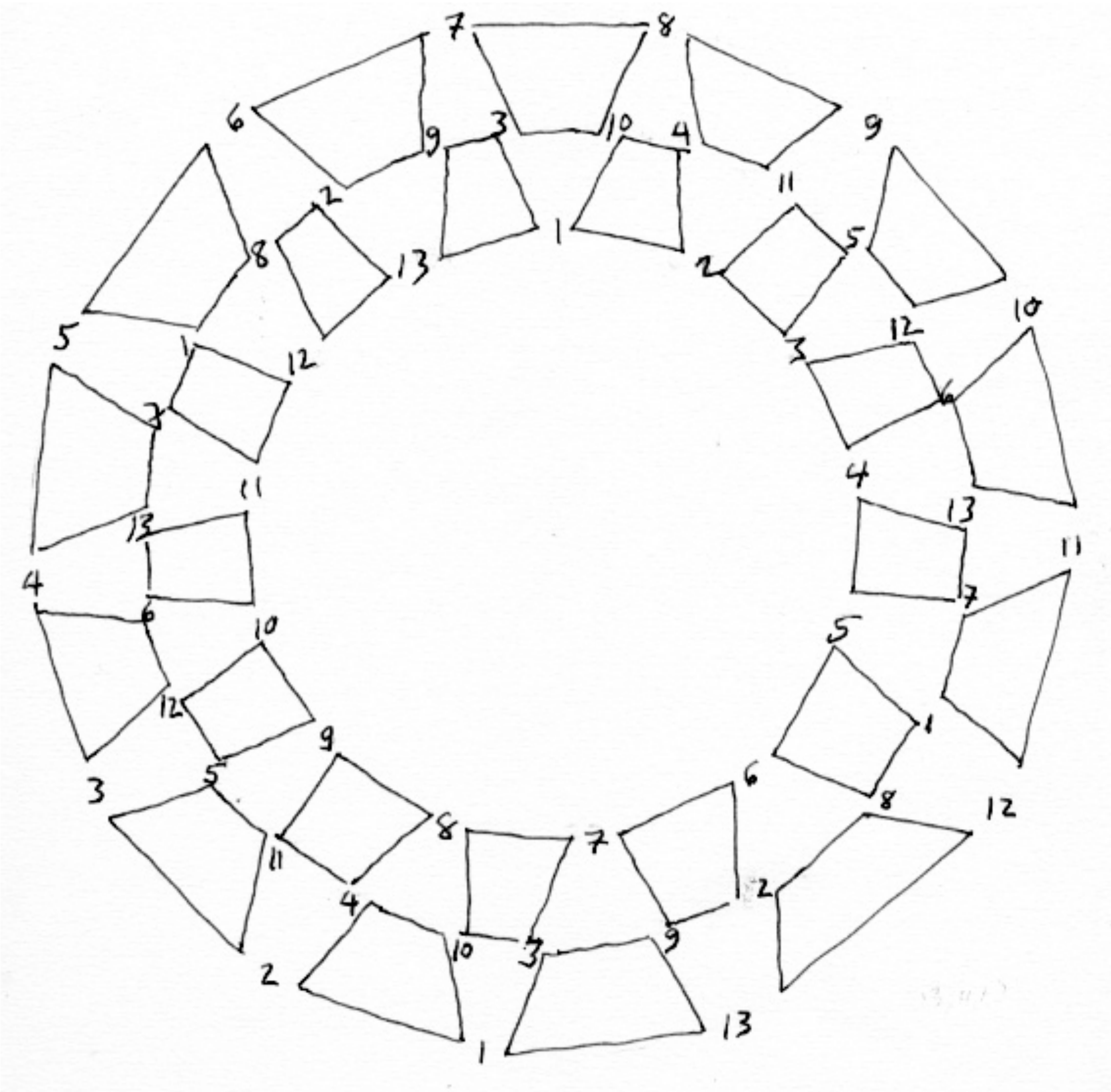
To hear this network, we can simply assign the numbers to a scale of seven notes, and read the white circle followed by the gray circle, as I have done in the music notation below. Of course the symmetries would be the same using any seven-note scale, but I rejected dozens of candidates before finding this one, where the notes and chords sound truly equal and the music seems to homogenize. I am going to play the 14 chords on the piano, but of course it is not really piano music, and could be heard equally well on another instrument (or instruments):



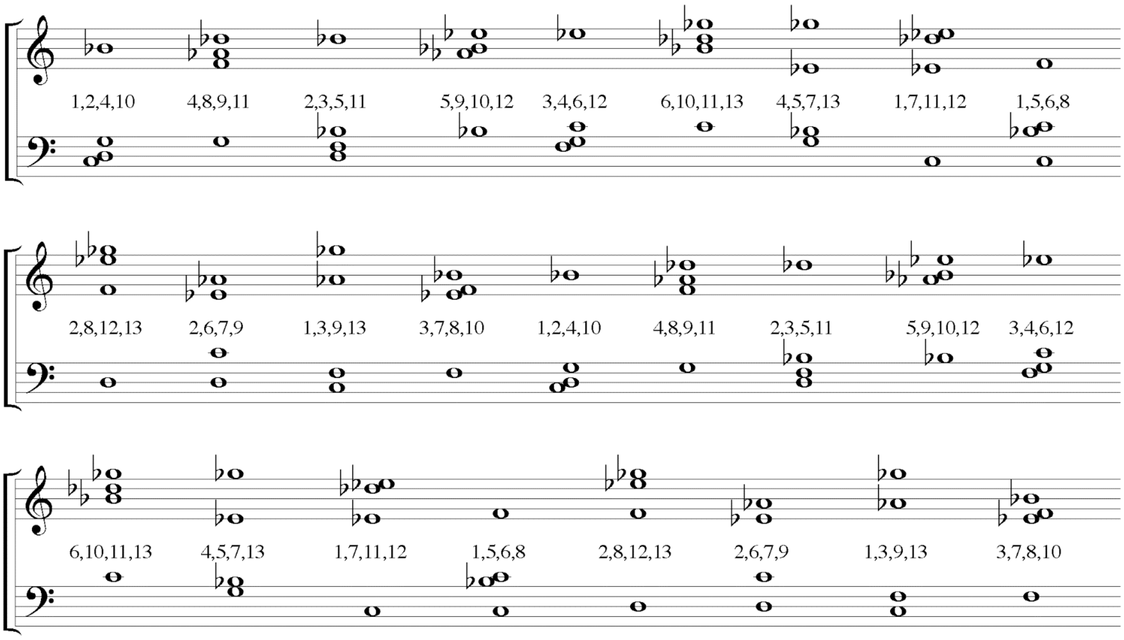
Could you hear the order? Could you hear that each note occurred the same number of times, that each pair of notes came together the same number of times, that everything is in perfect balance? Well, I must admit that I don't really hear this either, but it is remarkable how clearly one hears if there is a mistake, so the ear is somehow sensitive to what is going on. And of course, this is a new way of listening, and any new way of listening does require a bit of training. No doubt in a few years we will have learned how to perceive such patterns more easily.

Another question is whether we can call this a piece of music. I didn't really compose it. I simply found it within a mathematical phenomenon, and for some time I felt that such things were simply models or prototypes, rather than actual pieces of music. At the same time, when I try to expand the progression, to add a melody, to move the music through a series of variations, the result always seems vulgar. The sequence is much more satisfying in its natural state, and since the natural numbers really are a part of nature, this can be regarded as a little gem found in nature. It is quite lovely just as a diamond in the rough and does not need to be cut and polished.

Another block design with musical potential is (13,4,1), a collection of 13 four-note chords. In other cases there are many ways of forming a group of symmetrical subgroups, but in the case of (13,4,1) there is only one solution. Of course, you can always exchange the twos with the eights, for example, but this will just be a morphism of the basic block design. Again each note and each pair of notes occur the same number of times, and again it seems necessary to draw some pictures in order to uncover the nerve of the system. In this case the essence of the structure seems to emerge best in the following diagram. Let us listen to the sequence of two times 13 chords by alternating between the inner and the outer circles. You probably won't be able to count fast enough to be sure that all 13 notes occur the same number of times, and that the chords appear twice each, but I think you can hear that the music is somehow turning in a circle, and if I play a wrong note, you will hear that there was a mistake



(13,4,1) two circles



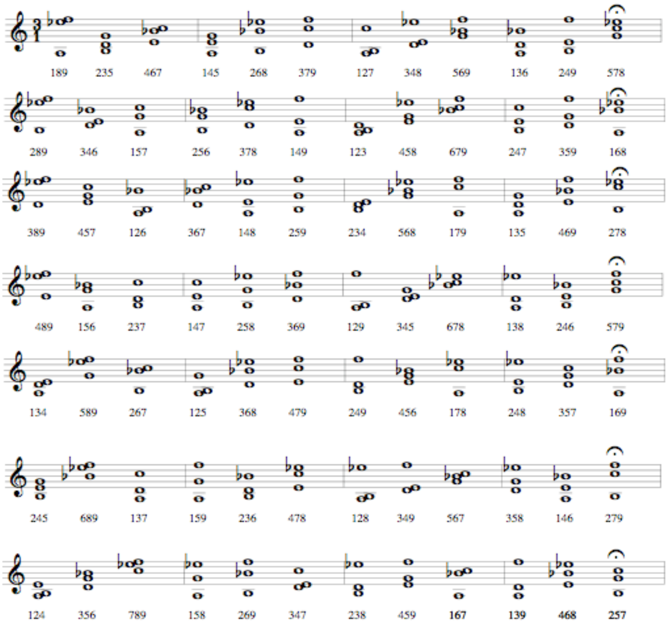
The comment offered by Dinitz and Archdeacon concerning (13,4,1) is probably too technical for most musician readers, but let me quote it for those who will understand:

This is the projective plane of order 3. It can be obtained from the (9,3,1)-design by adding 4 new points {∞_1 , ∞_2 ,∞_3 ,∞_4} and a new block containing them. Then to each block in the ith parallel class of the (9,3,1) design add the point ∞_i .

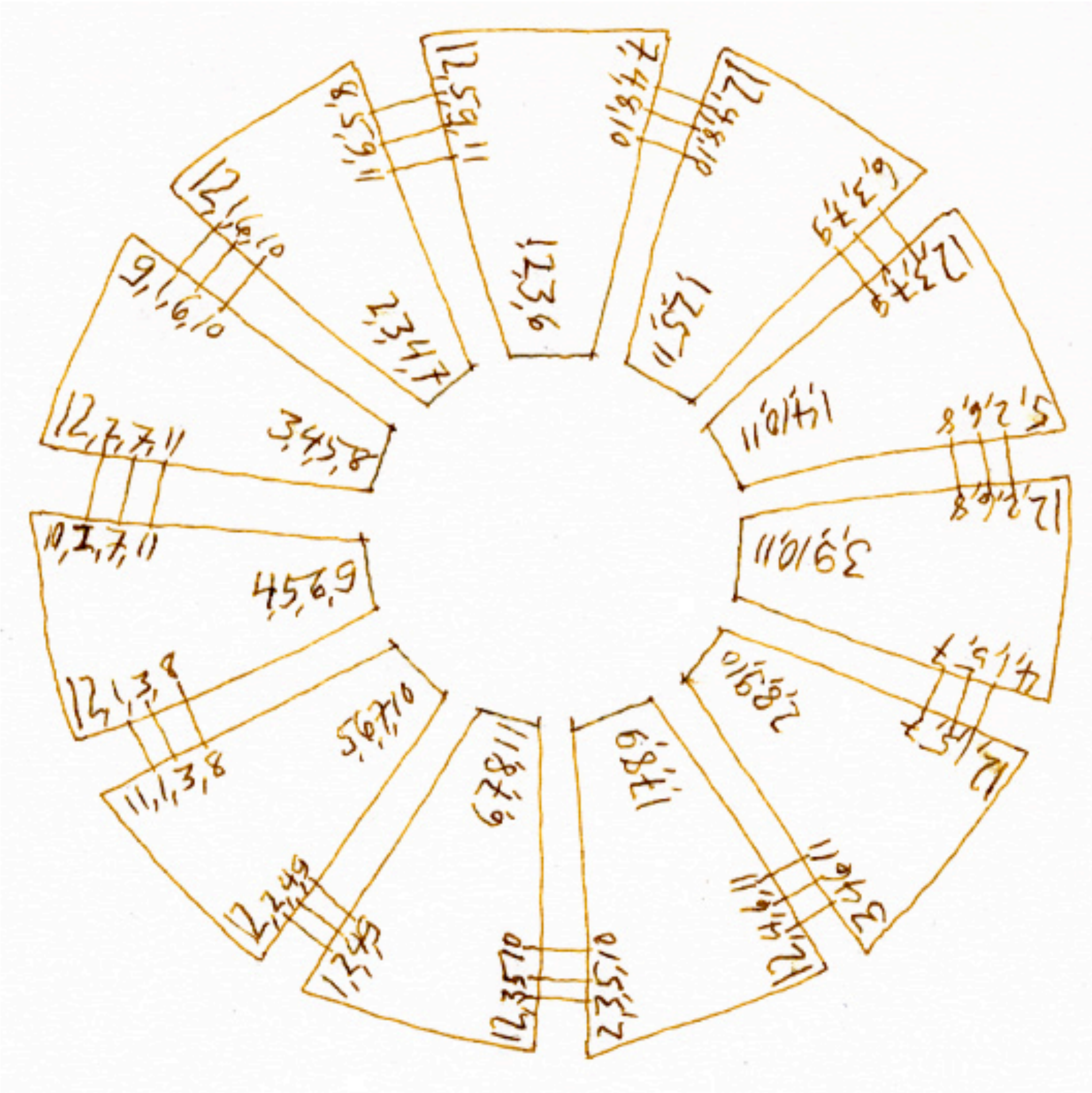
Again the resulting chord sequence is terribly short to be considered a musical composition, but it is a complete system nonetheless, and as I continue to study all these networks of chords, I have gradually concluded that such systems must be considered finished objects. If I try to “develop” the material the way composers are taught to do, I don’t really improve anything. Now, in August 2007, as I revise this text for the printed edition, I am also preparing an edition of about a dozen of these little *Networks.*

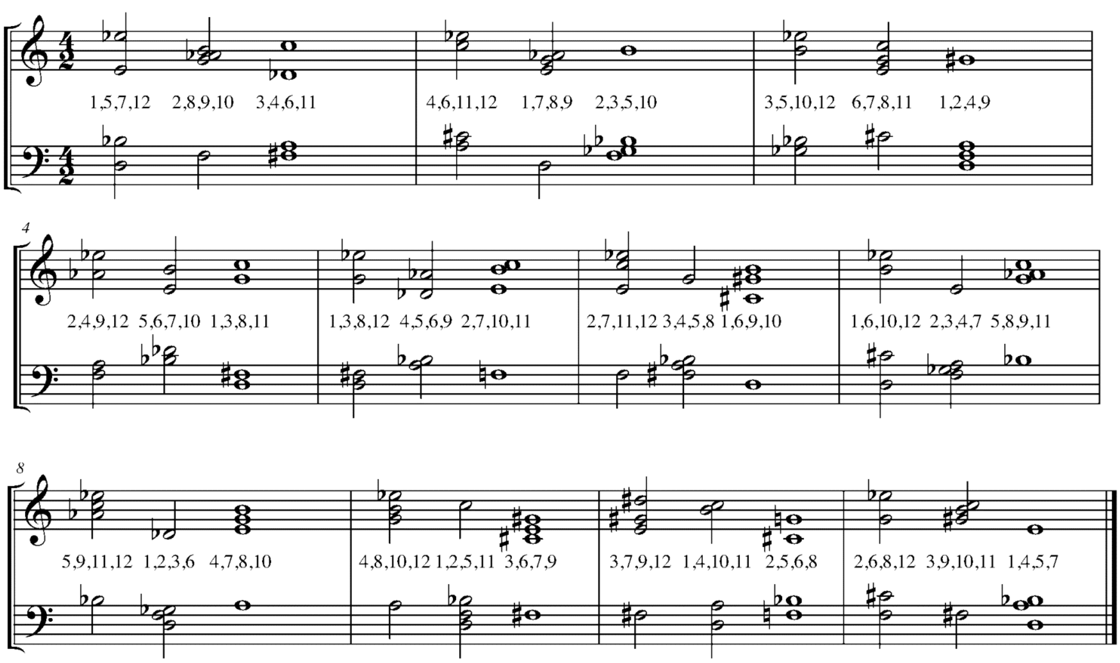
Of course, some combinatorial designs have more blocks and take more time. A good example is (9,3,1). The basic solution here consists of only 12 three-note chords, in which each note is used four times. But this group of 12 chords can be expanded to what the mathematicians call a large (9,3,1), combining seven different solutions to the problem. Now we have a system of 7 * 12 = 84 chords, in which all 84 combinations of the nine notes, taken three at a time, are included exactly once. That is difficult to draw, so I’ll just give you the music.

large (9,3,1)



I will not take the space here to include the three unique solutions of (10,4,2), but since we are so attached to our 12-tone chromatic tradition, I want to show you my realization of (12,4,3), so that you can see some 12-tone music that comes directly from a block design. Incidentally, Jeffrey Dinitz told me that mathematicians can prove the existence of 14 million non-isomorphic solutions for the (12,4,3) problem. Many of these are known as “resolvable,” which means that the 33 chords of this group can be divided into 11 groups of three chords, each of which includes all 12 notes. Here is the drawing and the music notation for my (12,4,3). It is resolvable, so each measure, each group of three chords, contains the complete scale:




1,5,7,12 2,8,9,10 3,4,6,11
4,6,11,12 1,7,8,9 2,3,5,10
3,5,10,12 6,7,8,11 1,2,4,9
4
2,4,9,12 5,6,7,10 1,3,8,11
1,3,8,12 4,5,6,9 2,7,10,11
2,7,11,12 3,4,5,8 1,6,9,10
1,6,10,12 2,3,4,7 5,8,9,11
8
5,9,11,12 1,2,3,6 4,7,8,10
4,8,10,12 1,2,5,11 3,6,7,9
3,7,9,12 1,4,10,11 2,5,6,8
2,6,8,12 3,9,10,11 1,4,5,7

It is also possible to form a large (15,3,1), which makes the 13 * 7 * 5 = 455 chords of *Kirkman's Ladies*, a 13 – minute piece that I wrote in 2005. Another block design, known as 4-(12,6,10) produced the 330 chords of *Block Design for Piano (2006)*, an 18-minute composition. Now I am completing a Septet, which transforms the 11 chords of (11,5,2) into 10 different solutions. And enough new possibilities arise that I expect to be exploring this area for some years to come.

This is of course only a miniscule introduction to block designs and to the chord groups one can find within this branch of combination theory. The definitive book on the subject, the *Handbook of Combinatorial Designs* (Chapman and Hall/CRC, second edition 2007), edited by Charles J. Colbourn and Jeffrey H. Dinitz, provides about a thousand pages of supplementary reading for those who wish to go further.