Musical Questions for Mathematicians

Tom Johnson

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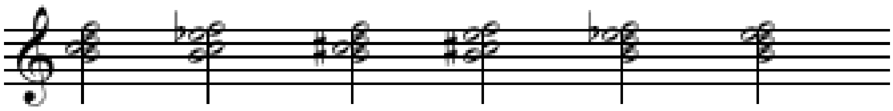
As a composer working with mathematical structures and deterministic sequences, I often confront questions I am not qualified to answer. At such times mathematician colleagues such as Jean-Paul Allouche or Franck Jedrzejewski or Emmanuel Amiot or Jon Wild are often able to give me answers, and sometimes this information enables me to compose a type of music that has never been heard before. Recently, working on a series of “rational harmonies,” I find that my progress is frequently blocked by mathematical barriers. Rather than bothering particular colleagues with my rather long list of rather thorny questions, it seems preferable to present them in a general way, so that mathematicians may work on whatever questions seem most interesting to them. Sometimes the answers they find may be useful in contexts far larger than some particular Tom Johnson composition.

I will limit this article to these current questions having to do with harmony, that is, with questions of how to construct chords logically and place them in natural groups and sequences. Such questions generally pertain to combination theory, but I can not solve them with the inclusion-exclusion theorem, by examining partitions, or with other simple strategies that I more or less understand. Naturally I will be extremely appreciative of any theorems that may result – even though my only formal training is in music and composition, and I probably won’t understand the proofs.

I will assume that the reader will not be confounded by terms like “C#” and “dominant seventh chord,” and has some idea of how harmonies may progress from one chord to another following some scale, but further musical sophistication should not be necessary. At the same time, I trust that the reader will tolerate my own limitations when I try to talk about mathematics

Same Adjacent Intervals

The six chords below have a similar sound, because the three adjacent intervals in all three cases are minor second, major second, and minor third.



but they are not considered equivalent in terms of traditional music theory, either tonbal or atonal.

In terms of tonal music theory the chords with C-sharp have little or nothing to do with the chords with E-flat or those with all white notes. If we wish to consider them in a more contemporary way and refer to Guerino Mazzola's chord categories, or to the affinity tables of Edmond Costère, or the vectors of Allen Forte, which are no doubt the best classification systems we have, the chords fall into three classes in another way. Only those which are exact inversions of one another are related. For my ears, however, there is a great similarity here, for the chords are simply six permutations of the same three intervals. They even form a mathematical group:

First Chord	(1,2,3)	neutral element
Second Chord	(1,3,2)	permutation (2,3)
Third Chord	(2,1,3)	permutation (1,2)
Fourth Chord	(2,3,1)	permutation (1,2,3)
Fifth Chord	(3,1,2)	permutation (1,3,2)
Sixth Chord	(3,2,1)	permutation {1,3}

The transpositions are the inverse of themselves, (1,2,3) is the inverse of (1,3,2), and one can make chains of logic without going beyond the four-chord vocabulary, as with other non-commutative groups

$$(1,2)(1,3,2) \neq (1,3,2)(1,2)$$
$$(1,2,3)\ (1,2,3)\ (1,2,3) = e$$
$$(1,3,2)\ (1,3,2) = (1,2,3)$$
$$(1,2,3)(1,2)(1,3)(1,3) = (1,3,2)$$
etc.

Do these six chords not form a musical category as well as a mathematical group?

How many such three-interval groups would we need to categorize all 495 four-note chords?

Chords with Common Tones

84 six-note chords may be formed with a nine-note scale. ${}_9C_6 = 84$. The seven chords below, all belonging to this group, have exactly four notes in common with each other chord. Notes 1 and 2 are present in all seven chords, but the other two common tones are different with each chord pair.

123689
124589
124679
123579
125678
123478
123456

Working with the same nine-note scale, might one find a group of 8 six-note chords having this property?

How large a group of six-note chords can be formed with this scale if it is required that each chord have exactly 5 notes in common with each other chord?

What groupings are possible if we enlarge the scale to 10 notes or reduce it to seven?

Could one answer all such questions with just a few coherent theorems?

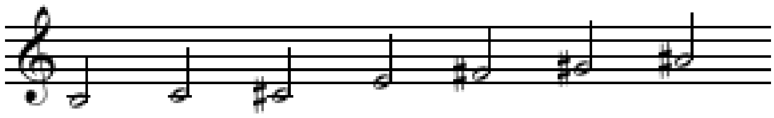
(In this case it is not at all clear to me what the musical value of such information might be, but perhaps if I understood the possibilities a bit better it would be easier to find interesting applications for particular cases.)

Finding the Smallest Scale

Guerino Mazzola lists 12 basic "chord categories" for three-note chords. All the other ${}_{12}C_3 = 220$ three-note chords are simply inversions, transpositions and cyclic displacements of these:

XXXOOOOOOOOO
XOXOOOOXOOOO
XXOXOOOOOOOO
XOXOOXOOOOOO
XXOOXOOOOOOO
XOOXOOOXOOOO
XXOOOXOOOOOO
XXOOOOXOOOOO
XOXOXOOOOOOO
XOXOOOXOOOOO
XOOXOOXOOOOO
XOOOXOOOXOOO

One can form all 12 with this seven-note scale, which might also be defined as (0,1,2,5,7,9,11)

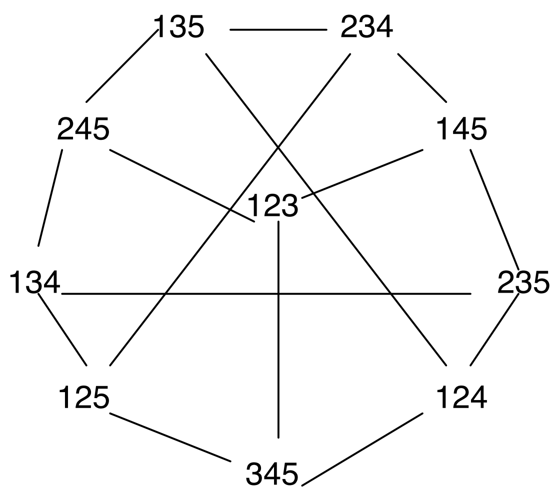


but a scale of six notes already contains 20 different three-note chords. Is it not possible to find the 12 basic trichords within some six-note combination?

What is the smallest scale that includes Mazzola's 29 categories of four-note chords?

Hamiltonian Circuits

A five-note scale contains 10 three-note chords. ${}_5C_3 = 10$. To have maximum harmonic movement, one might wish to require that there be two new notes as one passes from one chord to another. Guerino Mazzola told me that to answer such questions one must first observe the *nerve*, which means observing how everything is connected. In this case the possible connections can be drawn as a graph in two dimensions, and if one works long enough on the problem, one finds that the possible routes form this neat symmetrical pattern.



One can certainly not find a Eulerian path tracing all 15 lines, since three lines converge on each chord, but can one construct a Hamiltonian circuit, connecting the 10 chords in one loop?

What if we wanted each chord to have two tones in common with the one before and the one after?

What happens when we try to connect the 15 four-note chords that can be constructed from a six-note scale or the 21 five-note chords that can be constructed from a seven-note scale?

What general rules can be observed?

Finding the Chain

70 four-note chords may be formed using the notes of an eight-note scale. ${}_8C_4$ = 70. Each of the 70 chords has exactly two notes in common with 36 other chords. The chord (1,2,3,4), for example, connects with

1256	1257	1258	1267	1268	1278
1356	1357	1358	1367	1368	1378
1456	1457	1458	1467	1468	1478
2356	2357	2358	2367	2368	2378
2456	2457	2458	2467	2468	2478
3456	3457	3458	3467	3468	3478

There seem to be many ways to make a necklace of the 70 chords, where each chord has exactly two notes in common with the chord before and the chord after, but what is the best way to calculate such a sequence in a coherent mathematical way?

With a finite automaton?

With a symmetrical graph permitting a Hamiltonian circuit?

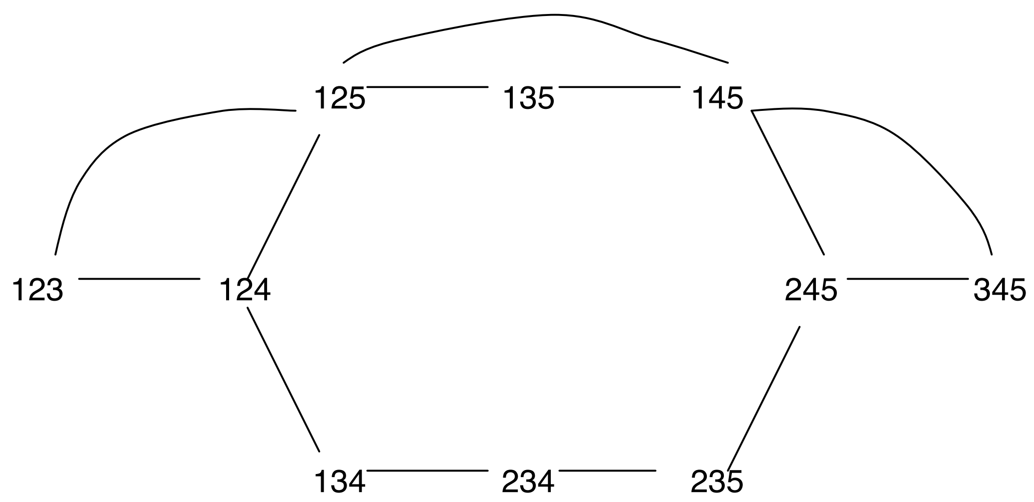
Through a series of systematic transpositions?

Is the most coherent mathematical sequence likely also to be the most coherent musical sequence?

One Voice Moving

Sometimes motions from one four-note chord to another are not very smooth, even when there are three common notes between the chords. Suppose that a chord consisting of scale degrees (1, 2, 3, 5) is followed by (1, 3, 4, 5). The third voice, the person playing note 2, now leaps over the second voice to note 4, and the ear hears that the harmony is jumping around. If we want only one voice to move from one note to some adjacent note, the restrictions are much greater than when we are satisfied simply to have a maximum of common tones, and unfortunately it is never possible to make a chain of a whole family of chords in this way.

The following nerve shows the 10 three-note chords that can be formed on a four-note scale and the ways in which they may be linked if one permits only one voice to move to one adjacent note. The curved lines indicate slightly less smooth connections. The voices do not cross, but the moving voice nevertheless jumps rather than moving to an adjacent note. Unfortunately, to move across the page one is required to use at least one of these curved-line connections.

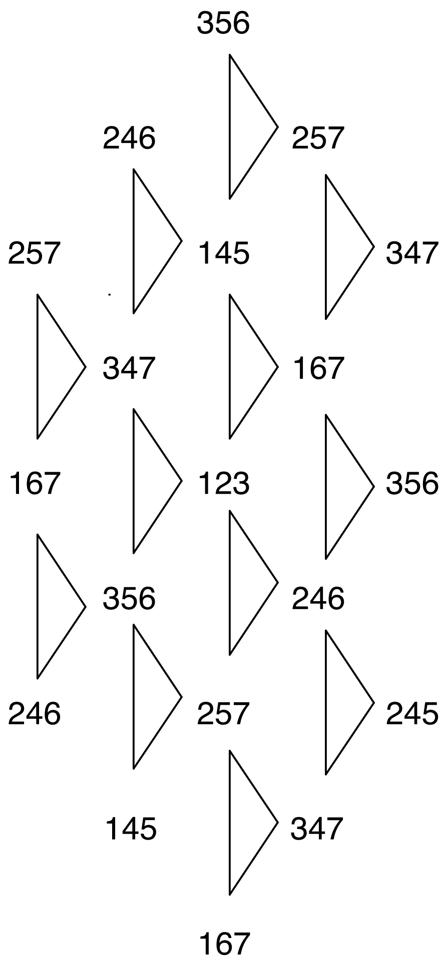


Will such limitations always occur with any group of x-note chords possible on a scale of y notes?

Would one be able to sequence chords more smoothly if one worked with other collections? The 24 chords containing the same 4 adjacent intervals for example? Or the 870 four-note chords having a total of 80?

A Steiner Triplet System

The seven circles in the diagram below mark the seven triplets of a Steiner triplet system, commonly defined as S(7,3,2). The digits 1 – 7 produce seven triplets, each of which has exactly one digit in common with each other triplet. Each of the seven digits can be found in three different triplets, as indicated by the handwritten numbers within the triangles.



Does this graph represent the true nerve of the system?

Is it a tiling that could be extended repetitively to tile an infinite plane?

Might it be a Penrose-type non-repeating tiling?

Or will it reach a dead end after a while?

Sums of Chords

I label notes chromatically from 5 to 36, permitting 32 possible notes, a range of two and a half octaves, and then ask the computer to calculate all the possible four-note combinations/chords that have particular sums. The largest categories are these:

A sum of 78: 865 chords
A sum of 79: 865 chords
A sum of 80: 870 chords
A sum of 81: 865 chords
A sum of 82: 865 chords

That is not very interesting as statistics, but as music it is rather significant, because we are looking here at four specific categories of chords, depending on whether their total, modulo four, is 0, 1, 2, or 3. A chord that can be spelled on the whole tone scale, for example, must consist of four even notes or four uneven notes, and thus can not appear among the groups with totals of 79 and 81. A basic dominant seventh chord, on the other hand, may have some transposition of the notes 0,4,7,10 or 1,5,8,11 or 0,2,6,9 or 0,3,5,9, or some other dominant arrangement, but the total will always be equivalent to 1 modulo 4. So dominant sevenths *can not occur* in four of the above groups, but will appear rather frequently among the 865 with sums of 81.

Is this a valid way, musically and mathematically, of classifying our 495 basic four-note chords?

How does this system for categorizing four-note chords differ from other better known systems of harmonic taxonomy?

Or is it all just one system, viewed from different angles?

What sorts of five-note chords might have the sum of 1 modulo 5?