Mathematical Rhythm, A Composer's Reaction to Recent Research

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Abstract

Twelve-tone theorists were primarily interested in understanding compositions stemming from the Viennese school and seldom regarded music in a purely formal way. They observed the all-interval tetrachord, however, and Babbitt proposed the hexachord theorem, and such things led gradually to the more general topic of homometric sets, where music theorists began to notice that all of this was applicable outside Z/12. This led to many new ideas that have been important for me as a composer, and I have written quite a few pieces involving particular sets of rhythms that would not have been conceivable without these theoretical discoveries. I explain a few of these here.

Precedents from 12-tone theory

Allen Forte was well aware of homometric relationships, which he referred to as "the Z relationship" in *The Structure of Atonal Music* (1973). He spoke of the all-interval tetrachords (0,1,3,7) and (0,1,4,6), and went on further to point out that three pairs of five-note chords and 15 pairs of six-note chords also have the Z relationship. At the same time he missed some important points. For example, he mentioned that "it is *perhaps not insignificant* that 6-Z/29 is the complement of 6-Z-50", while in fact we now know that this is completely necessary. Of course, Forte was interested in these hexachords primarily because he found a few of them in a Stravinsky score, and he wasn't really thinking about general structure and the music of the future. Still, his work reflected that of other researchers, like David Lewin, whose *Generalized Musical Intervals and Transformations* was also published in 1973, and which offered a complicated early proof of the hexachord theorem, showing that *any* subset of six notes has the same intervallic content as the other six notes of the chromatic scale. Shorter and more convincing proofs were later offered by J. E. Iglesias, Noam Elkies, Dmitri Tymoscko, Emmanuel Amiot, and Godfried T. Toussaint, and we now know that this can be generalized to sets of any even number when divided into two halves. With a length of 14, for example, a seven-note formation like (0,1,2,3,4,5,8), contains the same total distances between its elements as does its complement (0,1,2,3,4,6,7), with five distances of 1, three distances of 2, four distances of 3, two distances of 5, one distance of 6, and one of 7.

In that same period two other 12-tone theorists, Terry Winograd and Carleton Gamer, studied scales in which the distances between the notes each arrived in another quantity. They called these "deep scales" and this has led to "deep rhythms", which I will go into later. Jack Douthett and other theorists arrived at other important results by forming scales with “almost equal” distances, and all these discoveries are important today. The ideas of “deep” and “almost even” formations, for example, can now be generalized and applied to rhythm in a much broader manner.

Beyond Z/12

The big limitation with 12-tone theory was the idea of octave equivalence. If all octaves are equal, and one is working with pitches on the chromatic scale, then there is no reason to consider formations with more or less than 12 elements. Rhythmic theory had to be much more general because, even if one considers only regular metrical rhythms, they can come in any length.

These limitations are particularly clear if one considers **homometric** pairs. The idea of two subsets with the same total interval content was first proposed in 1940 by Lindso Patterson, who was a crystallographer and thus prepared to investigate crystals of any size, but for the music theorists of the 60s and 70s, only the size 12 cases noted by Forte seemed relevant. By now homometric formations of many sizes are often discussed, and Franck Jedrzejewski has calculated a complete list of homometric pairs (and triplets and more) all the way to cycles of length 24. These are numerous and the list comes to over 1000 pages.

Jack Douthett and others investigated **maximally even** scales, meaning scales that *almost* divided the octave into equal parts. Both the pentatonic and the diatonic scales do this, though for a long time no one thought about maximally even rhythms.

Terry Winograd and Carleton Gamer defined the pentatonic scale as **deep** because if one places the set (0,2,4,7,9) in a circle of 12 points and counts the 10 distances between the five notes modulo 12 one finds distances occurring four different times.

3 distances of 2 (0 to 2, 2 to 4, 7 to 9)
2 distances of 3 (4 to 7, 9 to 12)
1 distance of 4 (0 to 4)
4 distances of 5 (2 to 7, 2 to 9, 7 to12, 9 to 14)

They further observed that the 21 distances between the seven notes of the diatonic scale also give unique quantities, and this, like the idea of maximally even scales, seemed to be yet another justification for the universality of the traditional scales and a good reason for considering their formations “deep.”

Simha Arom was an ethnomusicologist, who observed what he called the r**hythmmic oddity**. He found in central Africa that rhythms in cycles of 16 hardly ever contain notes eight beats apart. The fact that the rhythms were never divided into two halves seemed to explain a lot about the syncopation of African music, and this is another observation that has become useful as one considers the theory of rhythm today.

Jon Wild of McGill University in Montreal has researched another category, which he calls **FLIDs**, that is, subsets with “flat interval distributions”. The all-interval tetrachords are FLIDs, because the interval distribution there is what Forte called (111111), with six interval occurring the same number of times. Such formations are rather rare, but Wild computed them all the way to the 15-note chords in cycles of 31. Curiously there are more in Z/31 than in any smaller cycles, and perhaps he or someone will explain why someday. FLIDs are generally rare, however. Wild sent me his compete list, all on one page.

I found it particularly interesting that a seven-note FLID and an eight-note FLID both occur in a cycle of length 15: (0,1, 2, 4, 5, 8,10) and (0,1, 2, 3, 5, 7, 8,11). The first formation has seven differences three times each and the second has seven differences four times each, and furthermore the two subsets are complementary. I don’t find any particular regularity or irregularity or other characteristics that distinguish these rhythms from rhythms that are not FLIDs. Wild wrote that he had been trying for years to define whether FLIDs are more or less predictable than other rhythms, or to find other universal FLID characteristics, but that he still hadn’t been able to do so either.

Other researchers might also be mentioned no doubt, but I will end with a reference to Godfried T. Toussaint’s *The Geometry of Musical Rhythm* (2013) which is the only book I know that brings together all of these things, and others.

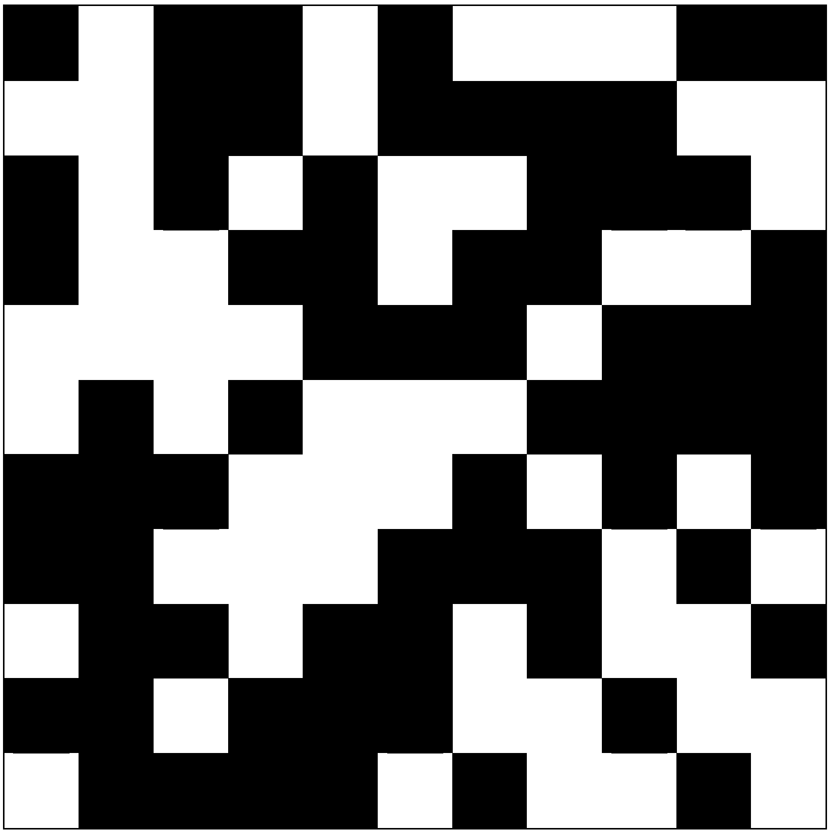
My own music was affected by all of this, as I had been looking for clear mathematical structures ever since the *Rational Melodies* (1982) and *Music for 88 (*1988). One thing that particularly interested me was putting together sets that are equal and complete, as I explain in the chapter “Equal and Complete” in *Other Harmony*, When everything has been included, when nothing is omitted and nothing is repeated, the structure is quite nice logically, and usually this logic can be heard as well.

Rhythms equivalent to harmonies

Jeffrey Dinitz, the co-author of *The Handbook of Combinatorial Rhythms*, had already been very helpful when I wrote to him with questions about block designs and related subjects, and he had even included a reference to my *Block Design for Piano* (2005) as an example of a 4-(12,6,10) block design in the latest edition of his *Handbook*. At one point I was interested in Room squares, first proposed by the Australian T. G. Room in 1955. I knew that a side-7 Room square is a square containing two-note subsets of elements 0 to 7 in such a way that all eight elements fall in each row and in each column, filling four of the seven columns and four of the seven rows, but I didn’t know how to put them together to make a set of rhythms, so I wrote to Dinitz with this question: Is it possible to construct five Room squares of side 7 such that each of the 35 rows has a different set of filled cells? A complete Room square might look like this, where the infinity sign represents element 7. Four different places are filled in each row and in each column. But how could this square be multiplied by five in order to have all 35 of the 7-choose-4 rhythms?

∞0			15		46	23
34	∞1			26		50
61	45	∞2			30	
	02	56	∞3			41
52		13	60	∞4		
	63		24	01	∞5	
		04		35	12	∞6

Finding the other four squares that complete this set of rhythms is pretty easy for a block design specialist, but after clarifying that, Dinitz and his bright graduate student, Susan Janiszewski, went on to calculate the Hadamard Room Squares of Side 11, which enabled them to deduce 11 by 11 squares. Incidentally, I am told that Hadamard matrices are of general interest to mathematicians in relation to Fugede's conjecture and many other mathematical questions:



This Hadamard matrix can be described also as an (11,6,3) block design, because the 11 elements are distributed into blocks of six elements, and each pair of elements comes together three times, both in the rows and in the columns, but for me it was a neat set of six-note rhythms. To make a Large (11, 6, 3) block design, containing all 462 11-choose-6 rhythms, the Vermont mathematicians had to calculate 42 such squares, but they somehow did it and sent me the numbers.

I was amazed at the mathematical beauty I saw when I looked at those 42 (11,6,3) squares (as a non-mathematician, I was tempted to call it a “miracle”), and I immediately began composing *Vermont Rhythms* (2006) for Klang, a fine sextet in The Hague. The mathematical logic was so clear that the music almost wrote itself. Composing mathematical music is often like that when the numbers are so clear that notes can be assigned to them in only one way, and sometimes the form and duration is also dictated. In this case the 42 solutions divided neatly into seven sections of six rotating textures in a 17-minute piece. Each section contained 11 measures of 11 beats, though I wrote the music in 12/8, with the twelfth beat always silent. To give a simple example, and avoid the confusion of all the transpositions, I will just show the piano part of one of the 11-bar sections. The chords are built on an invented 11-note scale, the rhythms fall on the first 11 beats of each measure, and the formation of the chords and the rhythms is always the same. I added the numbers of the beats and scale degrees for this explanation.

This was the first time I wrote a piece where the formation of the chords mirrored the formation of the rhythms, but the *Vermont Rhythms* sextet was followed by *Munich Rhythm*s for orchestra (2010), and *Dutch Rhythms* for two pianos (2018) which also mirror notes and rhythms. I want to show you also a bit of *Dutch Rhythms*, because the procedure is different in this case. Here the pitches and rhythms are calculated by permutations of twos and threes.

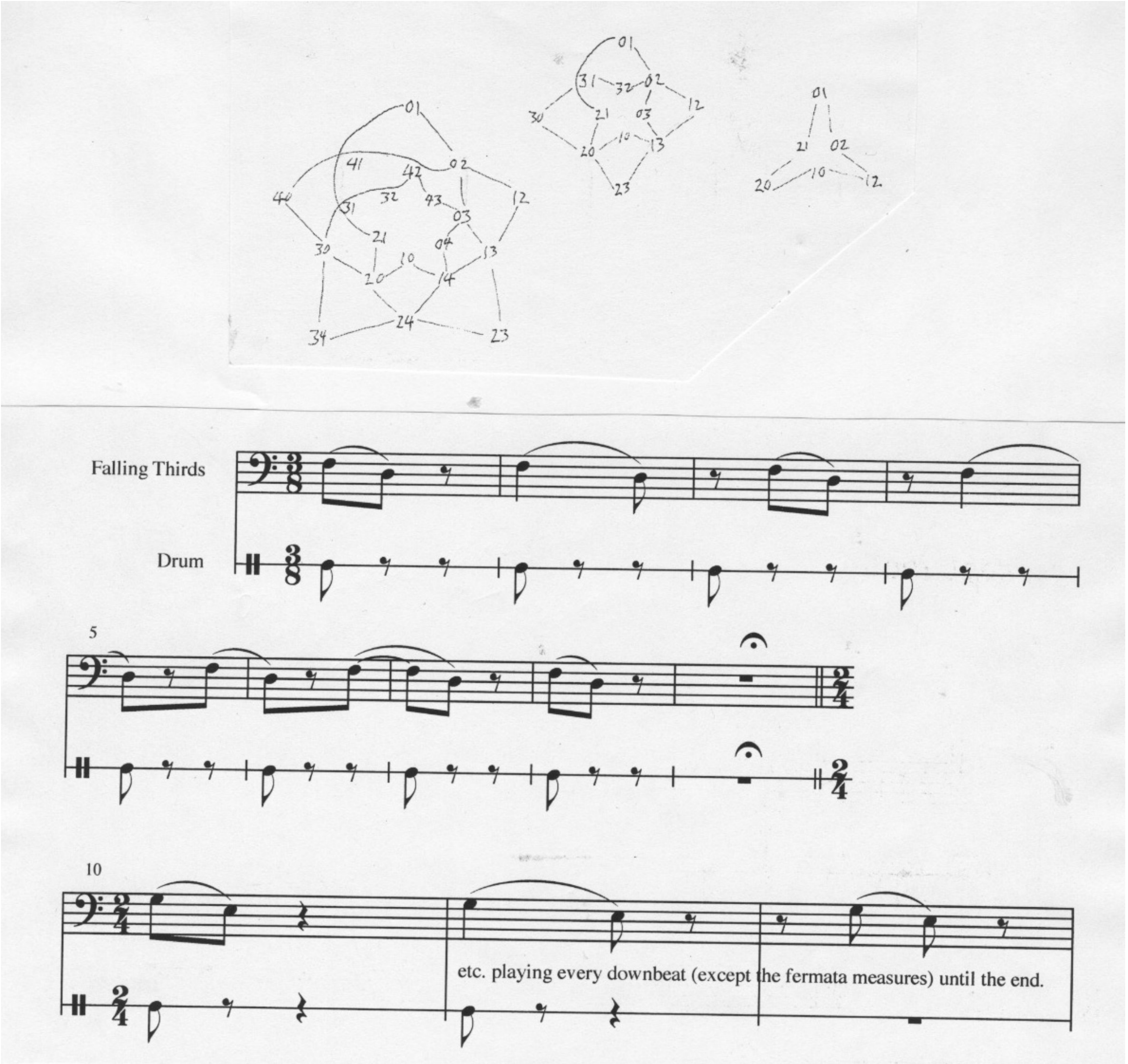
My teacher Morton Feldman said once “The history of music is the history of notation. Every new idea requires a new notation,” and a new notation seemed necessary here. Feldman saw notation changing a lot through all the graphic experiments and chance procedures with numbers and vague drawings and all, and it was especially clear that pieces like the Cage « variations », which had to be different every time, had to come out of some new kind of instructions. For me the story is much older than that. When somebody around 1600 wrote « vibrato » on a string part for the first time, that changed string music for three centuries. Then one day Bartok wrote « non vibrant » on a string part, and that was another new notation that made another new music. The problem of finding the right notation, like the problem of letting the music write itself, reflects something else that Feldman said very often. “Just let the music do what *it* wants to do.”

It can be difficult to read seven-note chords correctly when the differences are minimal, so I followed Feldman’s advbice and invented a new notation notating the rhythm of the repeated chords on the upper staff and the seven pitches of the chords on the lower staff. For this article I have also inserted the six digits of the permutations, indicating the intervals between the notes in ascending order in the lower system, the first piano, and in descending order in the upper system, the second piano. The rhythms come from the same set of permutations, with 2 corresponding to a note lasting one beat and 3 corresponding to a note lasting two beats. It is striking for me that one can hear a relationship between the tiny changes in the rhythm and the tiny changes that happen in the chords at the same time. Of course, sometimes one has palindromes like 322223 or 223322, in which case the two pianists play the same chords, and one can hear that too.

Rhythms from a graph

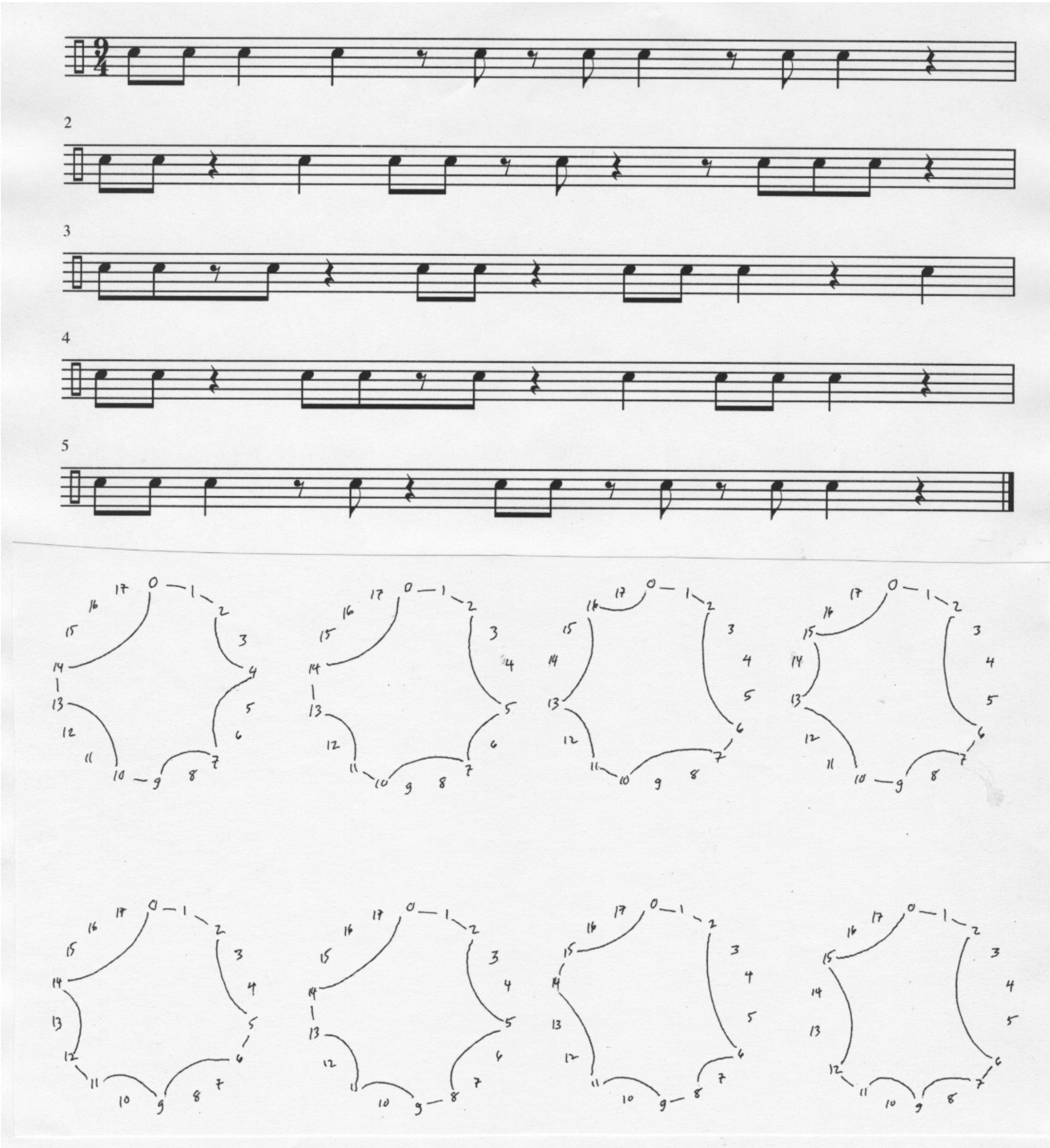
A piece called *Falling Thirds with Drums (2011)* expresses a collection of all the ways two beats can be connected around a circle. Consider a cycle of three beats (0,1,2) and play all the connections possible between two different beats, always following the cycle clockwise. Then continue with circles of four beats (0,1,2,3), five beats (0,1,2,3,4) and more, always following a systematic graph, where only one of the three beats changes with each combination.

I often calculate my music by drawing diagrams, and that was particularly true in this case. The piece continues with larger and larger configurations, ending with 10 elements after about 11 minutes, some solo instrument playing the descending thirds and the drum playing the zeros to orient what is happening. The drawing tracing the rhythms all the way to 10 is quite lovely, but too large to reproduce effectively here, so I will simply show the drawings I used to connect the unordered pairs in the cycles (0, 1, 2), (0 1, 2, 3), and (0, 1, 2, 3, 4) along with the first measures of the score. The (0, 1, 2) cycle, for example, begins with the third that begins on beat 0 and drops on beat 1, then the third that begins on 0 and drops on 2, then the third that begins on beat 1 and drops on beat 2, and so on, until all six unordered pairs have been heard.



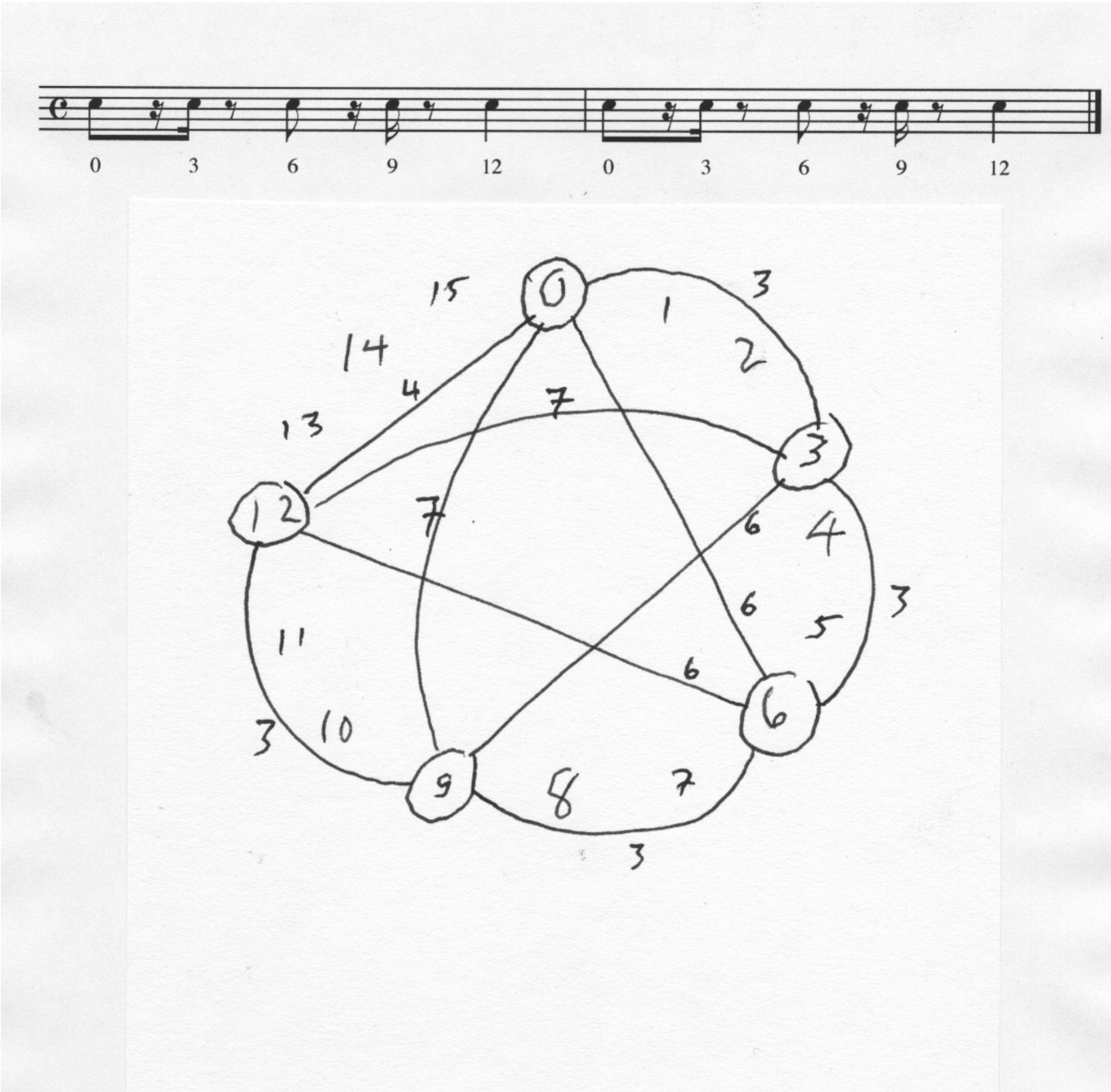
Rhythms from a homometric set

Another piece devoted to mathematical rhythms is a series called *Knock on Wood*, devoted exclusively to wooden sounds, and conceived at first as a sound installation containing several different elements. One of these elements or pieces is called *Knock on Wood: "Solution 571"*, because the piece comes directly from Jedrzejewski's list of homometries. It is the 571st of the 572 solutions of homometric subsets possible with nine-beat rhythms in an 18-beat cycle. In solution 571 four different combinations all have the same interval content. Two are complements of the other two, and all four can be read either clockwise or counter-clockwise, so I drew eight circles. It doesn't make sense to begin on a silence that can not be heard, and if one begins in the middle of sequence of four notes, one distorts the four adjacent notes into 1 plus 3 or 2 plus 2, so there are only five good starting points in each circle. If the eight circles are multiplied by five starting points and two directions we have a total of 80 rhythms, which is quite enough to make a modest wood block solo. Here we can see a few of the 80 rhythms, in no particular order, along with the drawings behind them. Note that in each case one finds three adjacent notes once, two adjacent notes twice, and two isolated notes. This is of course because they are all homometric, and the lengths of the rests between are also controlled by the homometrie. Despite the apparent difference between specific rhythms, one does hear a mathematical unity.



Deep Rhythms

The idea of deep rhythms began a long time ago, as mentioned before, but its general importance has become clear only recently, particularly in the book of Godfried T. Toussaint, also mentioned before. Taking this principle into the world of rhythm, Toussaint gives many examples of deep rhythms, particularly in cycles of 16 beats, and these rhythms are generally enticing, often reminiscent of salsa beats, and never something Beethoven would have written. One such example is (0,3,6,9,12), in which we find one distance of 4, two distances of 7, three distances of 6 and four distances of 3:



Note that this *deep* rhythm is also a good example of *almost even* and of the r*hythmic oddity*. These ideas are quite interrelated, though it is not quite sure why.

I wanted to find a *complete* list of deep rhythms, at least those occurring within cycles of 16 beats or less, and I thought that would be easy, but it wasn't. I was somewhat reassured when I wrote to Toussaint and found that he wasn't sure about the complete list either. I did find about 10 deep five-beat rhythms that I liked though, and that was enough to make part of a sound installation *Knock on Wood (2018).* Sometimes you just have to make do with incomplete sets.

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