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Combinatorial Designs in my Music

It must have been around 2003 when I heard a composition of Samuel Vriezen in Amsterdam. I was particularly interested in the harmony and asked him about the chords he used, and he told me it all had to do with 11 five-note chords, each of which has two notes in common with each other chord. It was hard to imagine that this was possible, and I determined to try to form such a group myself. After a couple of days of fruitless labor, I asked my mathematician friend Jean-Paul Allouche if he could help me. Well, he had never tried to solve problems like this himself, but he was pretty sure that I could find the answer if I went to the library and looked up "combinatorial designs" or "block designs". The next day I went to the Institute Henri Poincaré and found several references that launched me into an area of research that was to occupy me for several years.

The most important thing I found that day was the following problem, posed in 1847 by an English pastor and amateur mathematician named Thomas Pennington Kirkman:

> Fifteen young ladies in a school walk out three abreast for seven days in succession; it is required to arrange them daily so that no two shall walk twice abreast. (*Ladies and Gentleman's Diary*, Query VI, p.48)

Kirkman's ladies can be considered the beginning of combinatorial designs, which gradually become a serious subdivision of combination theory, and the Kirkman's ladies problem was soon to be identified as (15,3,1), meaning that 15 elements are distributed into subsets of three elements, with each pair coming together exactly once. 35 blocks are necessary to solve the problem: five blocks per day for a week of seven days.

The discussion quickly grew to include all sorts of investigations of symmetrical ways of constructing symmetrical sub-sets, and even the original 15-ladies problem did not end with Kirkman. Mathematicians began to wonder whether it would be possible for the ladies to continue their daily walks for a complete semester of 13 weeks, so as to include all 15-choose-3 = 455 possible three-lady combinations, once each, in what is now described as a Large (15,3,1) design. It was not until 1974 that computers were fast enough for R. H. F. Denniston of the University of Leicester to find a Large (15,3,1) design, and it was with his solution that I formed the 455 three-note chords in *Kirkman's Ladies.* Mathematicians have since counted a total of 80 non-isomorphic solutions to the basic (15,3,1) problem, but most of them do not permit us to arrange the 35 blocks in such a way that one can have all 15 ladies in five blocks every day. Here is a sample of the first two days on my 15-note scale:

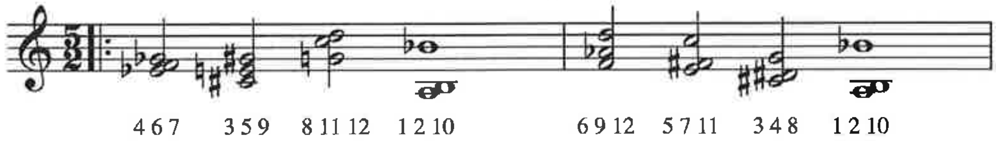
Kirkman's Ladies was finished in 2005, and for the next two years I studied smaller combinatorial designs, hoping to untwist their symmetries and find the beautiful music that I was sure was hidden there somewhere. I drew graphs, tried different chord sequences at the piano, chose different scales, started over, drew more graphs from other points of view, and gradually stumbled onto some systems that seemed better than the others. Sometimes a complete system consisted of only 10 or 15 chords, and the question arose: How could something so short be considered a musical composition? I sometimes tried to "develop" these harmonies, as composers have generally done, but every time I added a melody, or made an orchestration, or composed variations on a sequence, the result seemed vulgar.

More doubts arose when I listened to these sequences and asked myself: But am I really hearing this logic, really perceiving the symmetries of this network? Often I had to admit that I wasn't. It was just not possible to count all those notes and pairs of notes while listening. But I also noticed often that when I played a wrong note I would clearly hear a disturbance in the music. If some chord had only three notes instead of four, I would hear a flaw in the musical flow. In a sequence where the chords always had two notes in common with the preceding chord, and suddenly a chord arrived with only one common tone, the smooth progression was interrupted by a little jump. And in a context where every phrase contained a collection of nine notes, something sounded quite wrong if I played a phrase where one of the nine notes occurred twice and another was absent. If I was able to hear when things went wrong, I must also have been hearing the rightness when things were right.

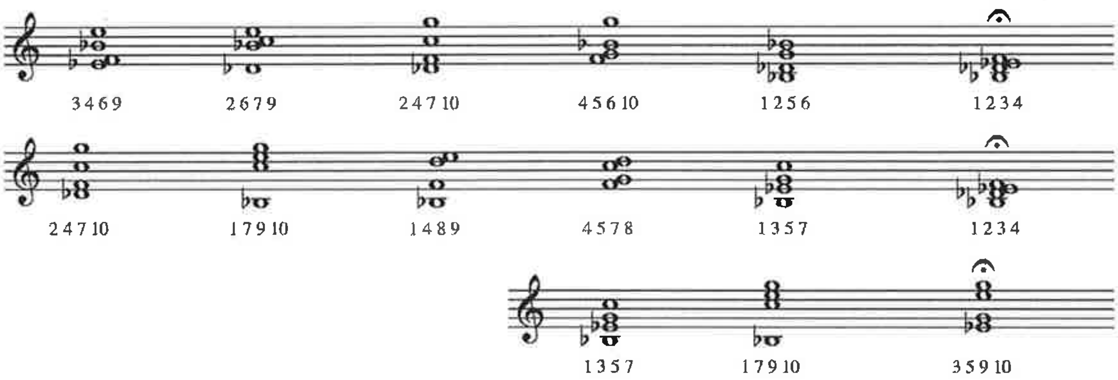
It would have been nice if all this had led to some grandiose statement of obvious importance, but composers should not try to tell their music what to do. The logic was pure and simple, and the music had to remain pure and simple as well. These short series of chords constituted complete systems and had to be left as they were. Whether something so short can be presented as a piece in a concert hall is another question, but at least two different ensembles are currently attempting to do this. My first presentation, however, was as an edition called *Networks*, with some notes by the mathematician Jeffrey Dinitz, presented at the Berlin colloquium of the newly formed Society of Mathematics, Music and Computation, in June 2007. Here are a few of the results:

(6,3,2)

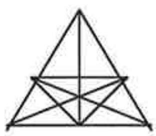
(12,3,2)



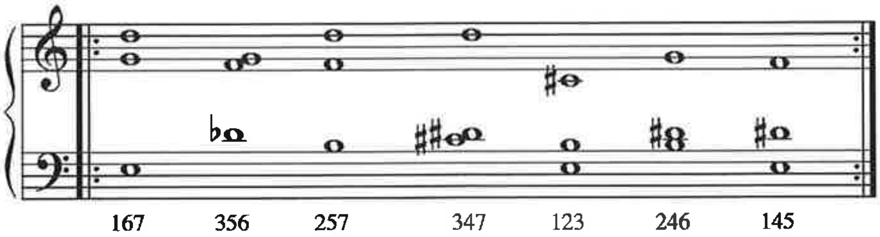
(10,4,2) solution 2



One design that particularly fascinated me actually predates Kirkman's researches, though it was first known primarily as the Fano plane, or a projective plane, rather than a (7,3,1) block design. Its simplicity and symmetry are quite evident if we look at it in geometric form.



The seven vertices are covered by six lines and one central triangle that is considered a seventh line. Each line connects three vertices, each vertice intersects three lines, each pair of points comes together on one line, each line shares one point with each other line, and this information can be summarized by saying simply that it is a (7,3,1) design. One can make combinatorial designs with almost any number of points, but it is rarely possible to represent these formations geometrically as neatly as one can here. I was convinced at that time that one could best hear and appreciate this formation if one transformed it into seven three-note chords, with just the right notes, so I spent many hours looking for the right notes, and finally ended up with this solution, which went into the Networks exhibition and edition.



This was a nice solution, neither too tonal nor too atonal, but for all of my work, it was an arbitrary choice. It was obvious that a number of other solutions could have been just as good.

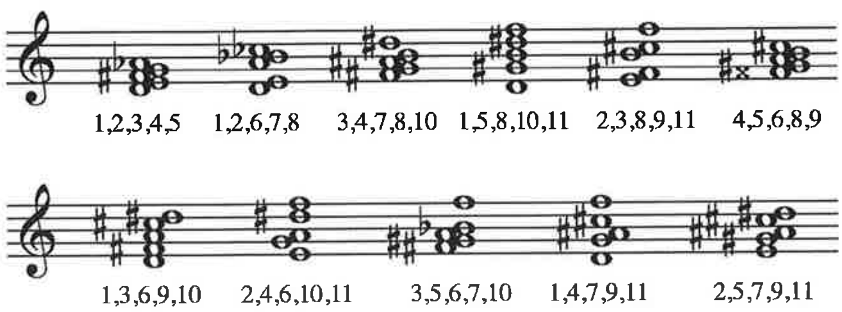
In 2009 I composed *Septet II* for 2 flutes, oboe, clarinet, 2 violins, and viola, and this composition is essentially a demonstration of my current opinion that the mathematics here is so strong that the seven chords will have a solid inter-dependent structure *regardless* of the notes chosen. Every phrase, every measure of this piece, is another (7,3,1) combinatorial design, and each one represents the same rigorous structure with all the notes and pairs of notes occurring equally. The music turns on 16 different scales, with the seven notes permuted eight times on each scale, and the orchestration of the seven instruments is constantly changing as well. By the end of the piece, lasting 12 minutes or so, we have heard 128 different (7,3,1) formations, only a fraction of the those possible, but enough to hear the mathematical symmetry that is common to all of them – and to countless others that are not heard. Here are the first four measures:

As I was calculating all this, I found that the results often surprised me. Each new set of seven chords and each new scale seemed like something I had never heard before. It was hard to believe that this was really my music, as it seemed to be coming from somewhere else, perhaps from that idealistic zone that Plato defined as *pure number*.

It was some years after my experience with the 11 chords, each having two notes in common with each of the others, that I came back to the configuration that Samuel Vriezen had used and employed it myself in *Septet I*. Eleven chords are constructed on an 11-note scale in a rather narrow range, following a combinatorial design known as (11,5,2), which means that:

> Eleven elements (11 notes) are distributed into 11 subsets of five elements (11 chords of five notes).
> Each note occurs five times in five of the chords.
> Each of the 55 pairs of notes comes together in two of the chords.
> Each chord has exactly two notes in common with each other chord.

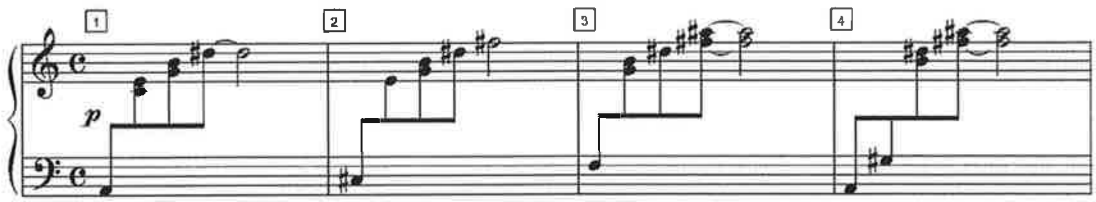
I simply took the unique solution for this rather amazing symmetrical structure, transformed it into 10 related solutions, selected my 11-note scale, and arranged the result for the seven instruments. Here are the 11 chords:



By now I have written a number of other pieces with combinatorial designs. Perhaps the most interesting mathematically is *Twelve (2008)* for piano, which uses just 12 of the more than 17 million non-isomorphic solutions for (12,4,3). I prefer not to discuss this piece here, as the 12 movements, and the 12 drawings accompanying them, all have different kinds of symmetries, and the piece should be treated by itself, preferably by someone else – particularly because it is in many ways a new kind of 12-tone music and needs to be regarded in context with the serial tradition. I can, however, give brief descriptions of some other pieces to give you some idea of the many ways in which one cn transform block designs into music.

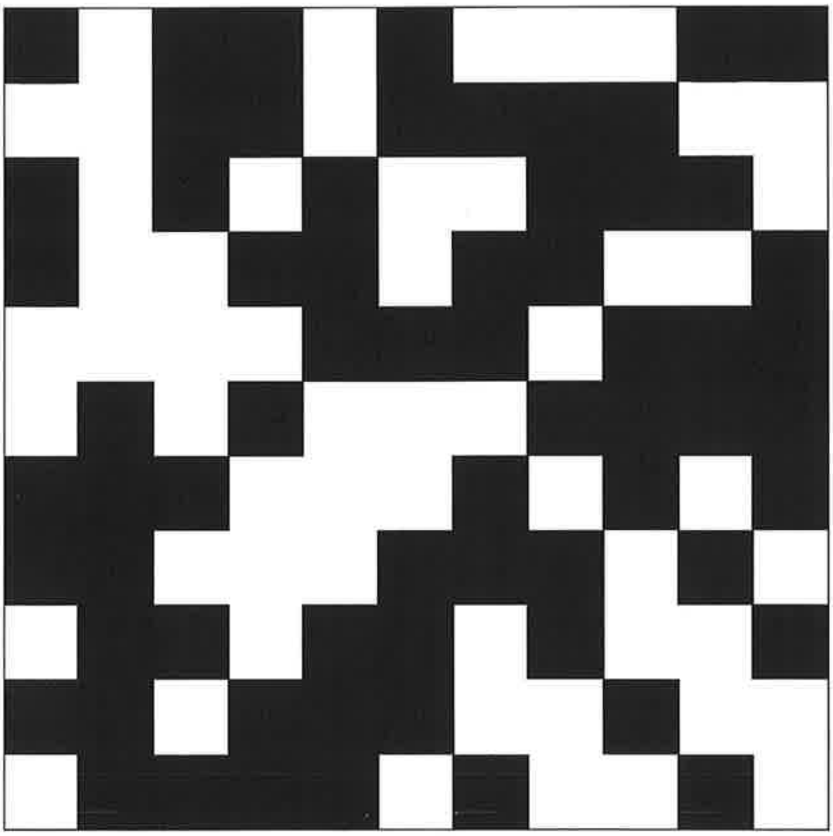
Block Design for Piano

Block Design for Piano employs a 4-(12,6,10) block design, which in my musical terms means that there are 12 notes, distributed into 6-note arpeggios, in such a way that every combination of four particular notes comes together exactly 10 times in 10 different arpeggios. In fact, it is also true that every combination of 3 notes comes together 30 times in 30 different arpeggios, every pair of notes occurs 75 times in 75 different arpeggios, and each of the 12 notes occurs 165 times, in exactly half of the 330 arpeggios. In a way, the piece is a total realization of Schoenberg's ideal, with exactly equal emphasis of each of the 12 notes. The music follows the numbers note for note, except that they occur two at a time when the difference between them is a major third. The first four bars, shown here, are missing the high D, which occurs in every measure during one half of the 10-bar phrases, and nowhere in the other half.



Vermont Rhythms

This piece is called *Vermont Rhythms* because it would never have been written without the cooperation of two Vermont mathematicians, working at the University of Vermont in Burlington, Susan Janiszewski, with her advisor, Professor Jeffrey H. Dinitz. I told them I was looking for a block design that would give me a complete set of the 11-choose-six rhythms in an interesting way, and they pointed out that the block design (11,6,3) was a good point of departure. Each solution of 11 blocks fills six squares horizontally and six vertically, each rhythm has exactly three beats in common with each of the 10 others, and you can see that clearly in this illustration:

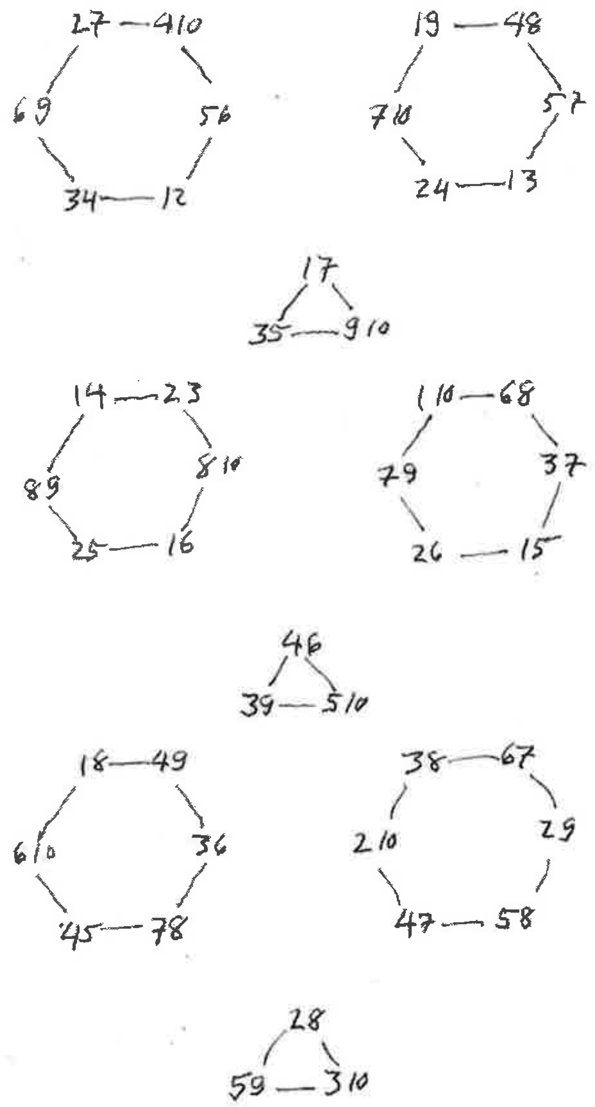


Janiszewski and Dinitz claimed they could put together 42 such solutions so that I would have 42 * 11 six-note rhythms, a complete set of the 11-choose-6 = 462 five-rhythms. Some months later they actually accomplished this, and I went to work for several months making a musical transformation. My primary interest was the 462 rhythms, but I soon realized that I could choose pitches by employing the 462 six-note chords possible on an 11-note scale at the same time, so I did that too. Of course, much of this organization will not be heard consciously, even by very astute listeners, but some of it will be quite clear to everyone, and it is satisfying to know that many unheard symmetries are also present, reflecting one another in the background.

Septet III

Septet III is a mechanical sounding little waltz that takes a long time to go no where in particular, but if you hear it, you will probably find that it does not *sound* at all trivial, and in fact, you will need time to enter into this little machine and begin to hear how it is working. The music follows an intricate system with notes spinning directly out of a Large (10,4,2).

A (10,4,2) design requires 15 blocks of four notes, and a Large (10,4,2) design, with all the 10-choose-4 = 210 blocks, requires a collection of 14 solutions. The construction I used was defined by Kramer, Magliveras and Stinson in 1991 in terms of permutations that I couldn't understand, so I am indebted to Franck Jedrzejewski, who calculated the 14 designs for me, following their description. Since each pair occurs twice, in two different blocks, one can describe the system by linking each of the 10-choose-2 = 45 pairs with the two other pairs that go with it to form four-note blocks. The visual result forms six hexagons and three triangles, as seen here.

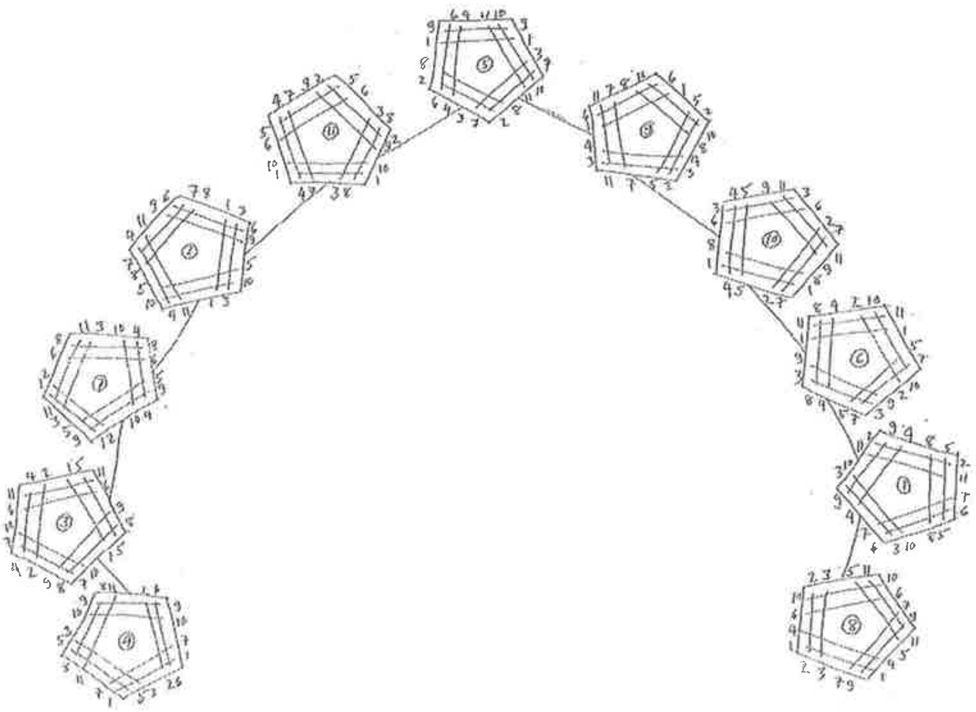


Each pair is shown only once, but each four-note chord is shown twice. (2,4,7,10), for example, occurs in the upper left hexagon as 2,7 – 4,10, in the upper right hexagon as 2,4 – 7,10, and in the lower right hexagon as 2,10 - 4,7. Except at the beginnings and ends of phrases, each measure contains two notes that are common to the preceding measure and two that are common to the following measure. Additional symmetries are so numerous that trying to find them all would be an endless task.

55 Chords for organ

I spent some months studying an amazing (11,4,6) design that had so many symmetries that my head was spinning day after day. For a long time I was only doing drawings and trying to connect and understand the 55 four-note chords of this formation, but eventually I managed to bring my findings together into a 23-minute piece for organ called *55 Chords.*

One remarkable thing I found was that the 55 chords could all fit together in the 11 symmetrical pentagons shown here. The circled number at the center of each pentagon indicates the one element not used in the five four-note chords represented on the edges of that pentagon. The lines inside the pentagons show how each chord has two notes in common with one of the other chords, and two notes in common with another. Adjacent chords have no notes in common, and there are no common tones as one moves from one pentagon to another.



Another formation I found rather amazing with the 55 chords formed by this (11,6,4) design is what I call "pairs of pairs." Four pairs of notes may be placed in a formation such that one reads one chord of the (11,4,6) design on the upper line, one on the lower line, one in the left column and one in the right column. One can connect all 55 chords in this way, but I will show only formations that include the chord (2,6,8,11):

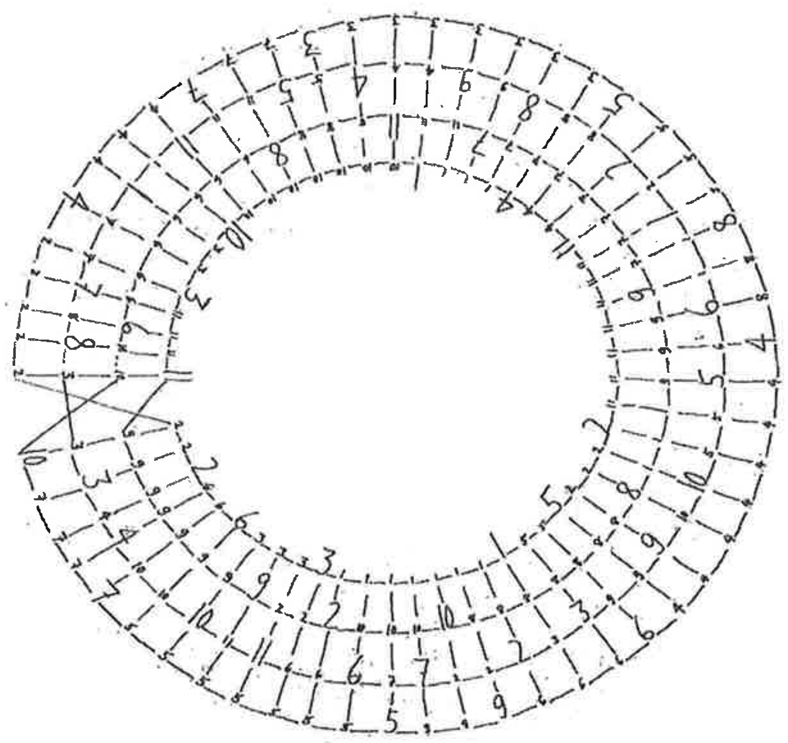
8,11	2,6
1,9	3,7

6,11	2,8
7,10	1,9

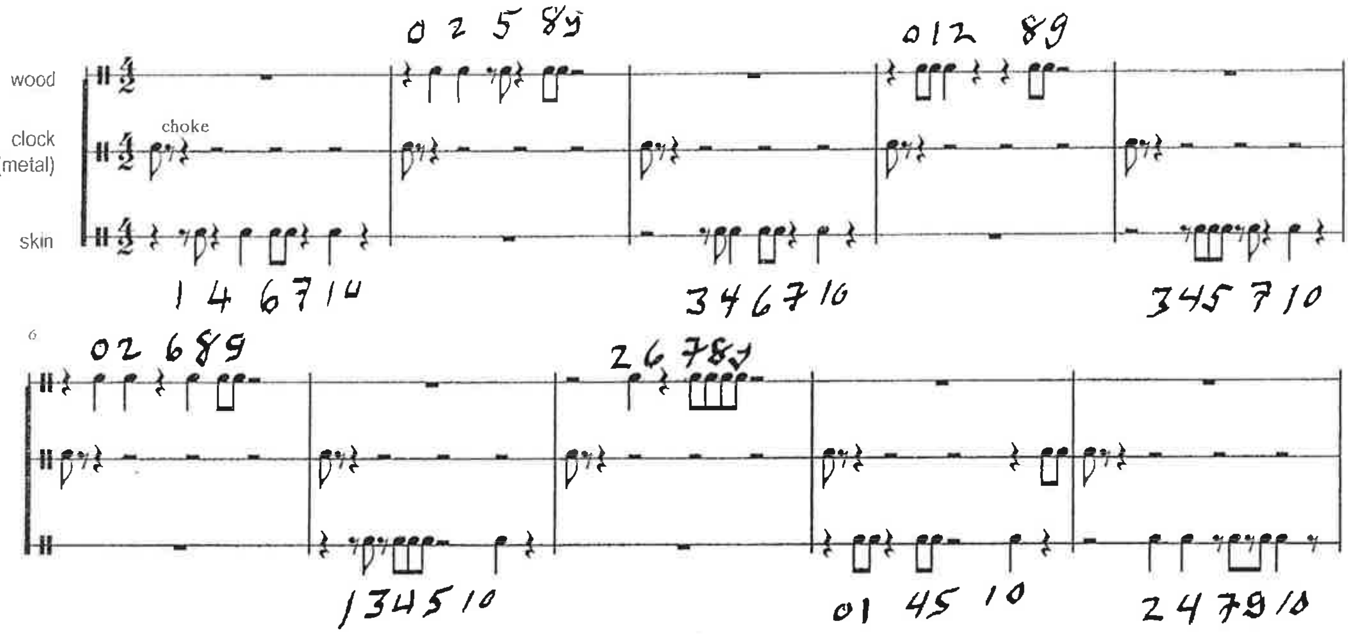
2,11	6,8
3,10	5,9

Dialogues for Percussion

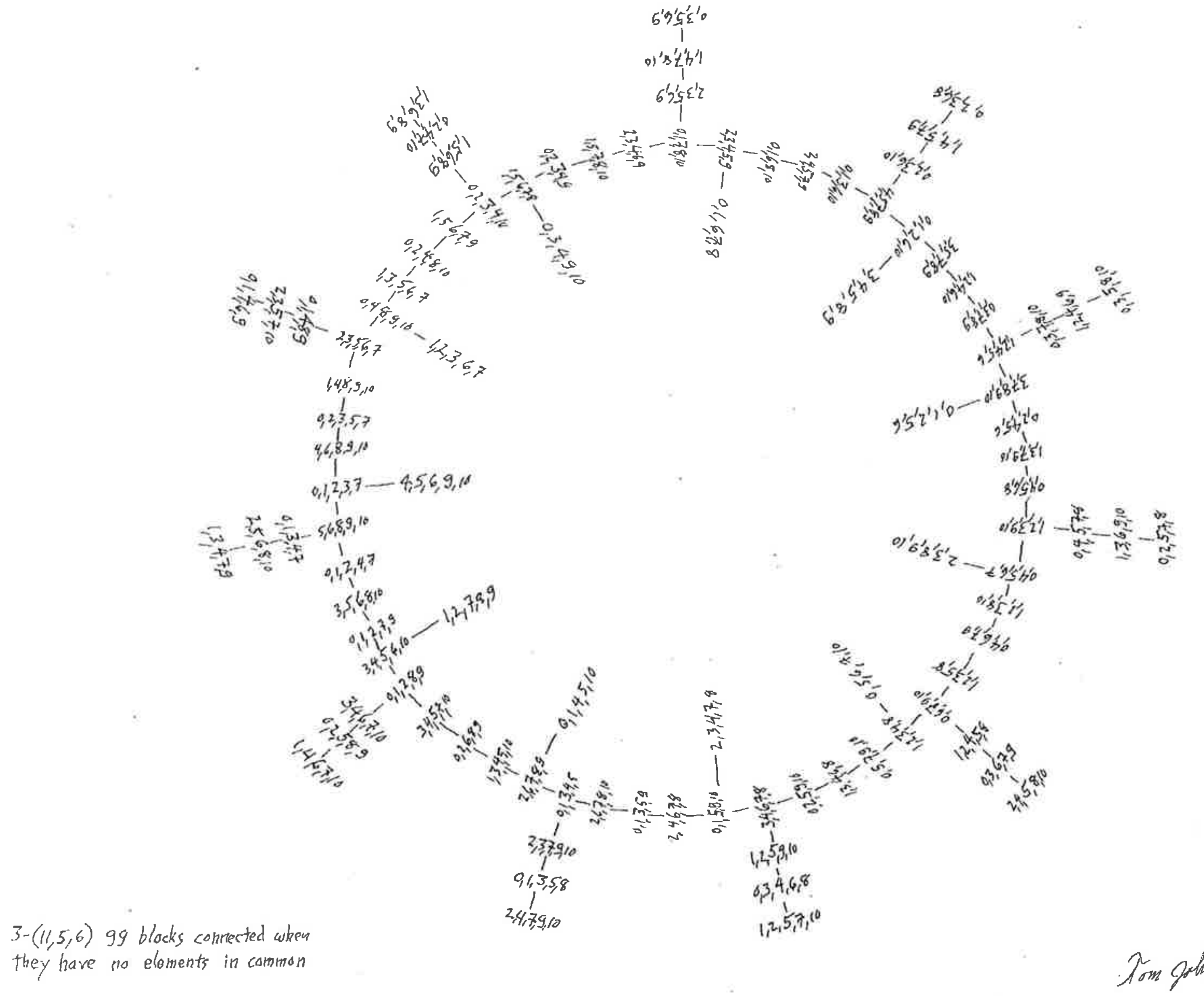
More recently I have been using block designs to organize rhythms rather than pitches, and two examples from *Dialogues* for percussion will demonstrate this. In one case I used the same design already used in *55 Chords* for organ, represented by this graph.



In the organ piece the music moved around the circle of 55 chords rather slowly, only one voice moving to form each new chord. In the percussion piece I followed the same mathematical logic, but instead of leaving one pattern and going immediately to the next, I inserted a measure containing the three beats that were common to the measures before and after. As the music moves, its changes are as subtle as with the harmonic changes, and perhaps more so.



Another movement of *Dialogues* comes from a 3-(11, 5, 6) design, with 99 blocks. Here the 3 indicates that each group of *three* elements comes together six times, not each pair. In this case each block has one, two or three connections with blocks that are completely different from it, as one can see in the drawing below. The 11 nine-measure phrases of the piece each begin with one of the outer rhythms and end with one of the inner rhythms. Thus each percussionist plays five-beat rhythms that have no similarities with the five-beat rhythms played by the other percussionist just before and just after. Of course, this also means that each rhythm is very similar to the rhythm two measures earlier and two measures later. The first few measures of the musical notation continue on the following page.



One could compose music of this sort without the help of mathematics, but one would never manage to treat all 11 beats exactly equally at the same time, and one would not even want to try see to it that each combination of three beats occurs exactly six times. Such symmetries can only be achieved with systematic intelligence, that is, with mathematics, and one could say the same about all the pieces discussed here.

References:

Colbourn, Charles J. and Dinitz, Jeffrey H., *Handbook of Combinatorial Designs,* second edition, 2007, Chapman and Hall. The standard and most complete book on the subject.

Johnson, Tom: All scores and many articles available at www.editions75.com